Lecture 14
Rotational Dynamics
Angular Momentum
Work and power
Angular momentum and torque

\[ \vec{\tau} = I \vec{\alpha} \]

But

\[ \alpha \equiv \frac{\Delta \omega}{\Delta \tau} \]

\[ \Rightarrow \tau = I \frac{\Delta \omega}{\Delta \tau} = \frac{\Delta (I \omega)}{\Delta \tau} \]

\[ \tau \equiv \frac{\Delta L}{\Delta \tau} \quad \text{and} \quad L \equiv I \omega \]
Angular Momentum

\[ \tau \equiv \frac{\Delta L}{\Delta t} \quad \text{and} \quad L \equiv I \omega \]

Angular momentum:

\[ L \equiv I \omega = (mr^2)\left(\frac{v_{\tan}}{r}\right) = rp_{\tan} = mr v \sin \theta \]

Just as linear momentum quantifies the amount of motion, angular momentum quantifies the amount of rotation. Note that as with torque, the value depends on the center used.

Units: \( \text{kg m}^2/\text{s} = \text{N m s} = \text{J s} \)
Energy of Rotational Motion of system of particles

Consider a rigid body being rotated around an axis perpendicular to the page and through point P. The body can be broken down into n particles.

Moment of inertia (about this axis): 

\[ I = \sum m_i r_i^2 \]

\[ KE_{rot} = \frac{1}{2} I \omega^2 \]

Perpendicular distance between the axis and the particle:

\[ v_i = \omega r_i \]
Angular momentum for rotating system

Consider a rigid body being rotated around an axis perpendicular to the page and through point P. The body can be broken down into n particles.

\[ L = \sum_i L_i = \sum_i r_i \quad p_{i,\text{tan}} = \sum_i r_i m_i v_{i,\text{tan}} = \sum_i m_i (r_i \omega r_i) \]

\[ L = \left\{ \sum_i m_i r_i^2 \right\} \quad \omega = I_{\text{tot}} \vec{\omega} \]

Total Angular momentum
Conservation of angular momentum

The (usually) most interesting situation:

When the \textit{net} torque on a system is zero, the total angular momentum of the system is conserved.

\[ \tau_{\text{net}} = 0 \implies \frac{\Delta L}{\Delta t} = 0 \implies L_i = L_f \]

Recall:

\[ F_{\text{net}} = 0 \implies \frac{\Delta p}{\Delta t} = 0 \implies p_i = p_f \]
Work and power done by a torque
(for pure rotations)

A force \( F \) acts on an object as it rotates from \( \theta_1 \) to \( \theta_2 \).

Work done along a small displacement:

\[
W = F_{\tan} \Delta s = F_{\tan} r \Delta \theta = \tau \Delta \theta
\]

Average power:

\[
\bar{P} = \frac{W}{\Delta t} = \frac{\tau \omega}{\Delta t}
\]

Instantaneous power:

\[
P = \lim_{\Delta t \to 0} \frac{W}{\Delta t} = \tau \omega
\]
equivalence between translational linear motion and rotational motion

\[ \theta \iff x \quad W = \tau \, \Delta \theta \iff \Delta W = F \, \Delta x \]

\[ \omega \iff v \quad KE = \frac{1}{2} I \omega^2 \iff KE = \frac{1}{2} m v^2 \]

\[ \alpha \iff a \quad P = \tau \, \omega \iff P = F v \]

\[ I \iff m \quad L = I \omega \iff p = m v \]

\[ \tau_{\text{net}} = \frac{\Delta L}{\Delta t} \iff F_{\text{net}} = \frac{\Delta p}{\Delta t} \]

\[ \tau_{\text{net}} = I \, \alpha \iff F_{\text{net}} = m \, a \]
A spherical shell rotates about an axis through its center of mass. It has an initial radius $R_i$ and angular speed $\omega_i$. By applying a radial force, we can cause the sphere to collapse to $R_f = R_i/3$. What is the ratio of the final and the initial angular speed, $\omega_f/\omega_i$?

A. $1/9$  B. $1/3$  C. 1  D. 3  E. 9
Example: Collapsing sphere

A spherical shell rotates about an axis through its center of mass. It has an initial radius $R_i$ and angular speed $\omega_i$. By applying a radial force, we can cause the sphere to collapse to $R_f = R_i / 3$. What is the ratio of the final and the initial angular speed, $\omega_f/\omega_i$?

A. 1/9  B. 1/3  C. 1  D. 3  E. 9

The force to collapse the sphere is radial, so it produces zero torque.

Net torque = 0  $\rightarrow$  Angular momentum is conserved.

$$L_i = I_i \omega_i \quad L_f = I_f \omega_f$$

$$I_i \omega_i = I_f \omega_f \quad \frac{\omega_f}{\omega_i} = \frac{I_i}{I_f} = \frac{MR_i^2}{MR_f^2} = \frac{R_i^2}{R_f^2} = 9$$

DEMOs: Hoberman sphere
Example: Diver

Weight does not produce any torque about the CM of the diver

$\rightarrow \quad L$ is conserved

Small $I$, large $\omega$

Large $I$, small $\omega$
ACT: Conservation of L

A student sits on a rotating stool and holds a rotating horizontal bicycle wheel by a rod through its axis. The stool is initially at rest. The student flips the axis of rotation of the wheel by 180°. What happens to the stool?

A. It rotates in the same direction as the wheel after the flip.

B. It rotates in the same direction as the wheel before the flip.

C. Nothing! Why would it rotate at all?
DEMOs: Chair and bicycle wheel

No change in $L_{\text{total}}$

$L_{\text{total}}$  

$L_{w}$  

$L_{S+S}$  

$L_{w}$  

$L_{\text{total}}$
Both systems can pivot about one end, and they are released from the horizontal position. Use conservation of energy to determine which system is moving faster when it reaches the vertical position.

1: Uniform rod of mass $M$

2: Massless rod and ball of mass $M$

A. System 1  
B. System 2  
C. Same for both
Both systems can pivot about one end, and they are released from the horizontal position. Use conservation of energy to determine which system is moving faster when it reaches the vertical position.

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A. System 1  B. System 2  C. Same for both
1: Uniform rod of mass $M$

$$I_1 = \frac{1}{3} ML^2$$

Same gravitational potential difference (same $\Delta h$)

2: Massless rod and ball of mass $M$

$$I_2 = M \left( \frac{L}{2} \right)^2 = \frac{1}{4} ML^2$$

Same final

$$KE = \frac{1}{2} I \omega^2$$

$$I_1 > I_2$$

$$\omega_1 < \omega_2$$
EXAMPLE: Falling tree

What is the angular acceleration of a tree as it falls down?

Model of a tree: uniform rod of length $L$
Newton’s 2nd law for rotations:

\[ \tau_{\text{net}} = I \alpha \]

\[
mg L \cos \theta = \left( \frac{1}{3} mL^2 \right) \alpha \quad \Rightarrow \quad \alpha = \frac{3g}{2L} \cos \theta
\]
EXAMPLE: Yo-yo

A cylinder with a massless string wrapped around it is released as the free end of the string is kept fixed. Find the acceleration of the cylinder.
Newton’s 2nd law for the translation of the CM:

\[ mg - T = ma \]

Newton’s 2nd law for the rotation about the CM:

\[ TR + mg \times O = I \alpha = \frac{1}{2} mR^2 \alpha \]

String does not slip on the cylinder:

\[ a = \alpha R \]
Note that $a < g$: The net force on the cylinder is less than $mg$. 
ACT: Real Atwood’s machine

Compare the tensions on each side of the rope.

A. $T_1 < T_2$
B. $T_1 = T_2$
C. $T_1 > T_2$
ACT: Real Atwood’s machine

Compare the tensions on each side of the rope.

A. $T_1 < T_2$
B. $T_1 = T_2$
C. $T_1 > T_2$
Example: Real pulley

Two 3.0-kg boxes is attached to a light rope that passes over a pulley (uniform disk of 2.0 kg, radius 20 cm) as shown below. There is no friction between the pulley and its axis or between the box and the table. The system is released from rest. Find the speed of box B when it has fallen 1.5 m.
Two 3.0-kg boxes is attached to a light rope that passes over a pulley (uniform disk of 2.0 kg, radius 20 cm) as shown below. There is no friction between the pulley and its axis or between the box and the table. The system is released from rest. Find the speed of box B when it has fallen 1.5 m.

The net work is due to the weight of box B, so $E$ is conserved.

$mg(h + 0) = 0 + KE_{trans,A} + KE_{trans,B} + KE_{rot,pulley}$

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$

Two unknowns $(v, \omega)!!$
If the string does not slip on the pulley, the blue point (on the edge of the pulley) and the red point (on the string) move together:

\[ v_{\text{red}} = v \text{ (of the boxes)} \]

\[ v_{\text{blue}} = R\omega \]

No slipping: \[ v = R\omega \]
\[ mgh + 0 = 0 + KE_{\text{trans},A} + KE_{\text{trans},B} + KE_{\text{rot},\text{pulley}} \]

\[ mgh = \frac{1}{2} m v^2 + \frac{1}{2} M v^2 + \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \omega^2 \]

no slipping: \( R \omega = v \)

\[ mgh = m v^2 + \frac{1}{4} M v^2 \]

\[ v = \sqrt{\frac{4m}{4m + M} gh} = \sqrt{\frac{4(3.0 \text{ kg})}{4(3.0 \text{ kg}) + 2.0 \text{ kg}} (9.8 \text{ m/s}^2)(1.5 \text{ m})} = 3.5 \text{ m/s} \]

\( m = 3.0 \text{ kg} \)
\( M = 2.0 \text{ kg} \)
\( R = 20 \text{ cm} \)
\( h = 1.5 \text{ m} \)
EXAMPLE: Cylinder rolling down an incline

A cylinder of mass $M$ and radius $R$ rolls down an incline of angle $\theta$ with the horizontal.

If the cylinder rolls without slipping, what is its acceleration?
Newton's 2nd law for translation of the CM:

\[ M g \sin \theta - f_s = M a_{CM} \]

Newton's 2\textsuperscript{nd} law for rotation about the CM

\[ \tau_{\text{net}} = \sum F_i r_{i\perp} \]

\[ I \alpha = f_s R \quad \text{Rolling without slipping} \]

\[ \Rightarrow a_{CM} = R \alpha \]
Newton's 2nd law for translation of the CM:

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Newton's 2^{nd} law for rotation about the CM:

\[ \tau_{net} = \sum F_i r_{i\perp} \]

\[ I \alpha = f_s R \]

Rolling without slipping:

\[ \Rightarrow a_{CM} = R \alpha \]

Three equations and three unknowns: \( \alpha \), \( a_{CM} \) and \( f_s \)
\[
\begin{cases}
Mg \sin \theta - f_s = Ma_{CM} \\
f_s R = I_{CM} \left( \frac{a_{CM}}{R} \right)
\end{cases}
\]

\[\Rightarrow Mg \sin \theta - Ma_{CM} = I_{CM} \frac{a_{CM}}{R^2}\]

\[a_{CM} = \frac{gsin\theta}{1 + \frac{I_{CM}}{MR^2}} = \frac{gsin\theta}{1 + \frac{1}{2}} = \frac{2}{3}gsin\theta\]

\(I_{\text{solid cylinder}} = \frac{1}{2}MR^2\)

(compare to \(gsin\theta\), the results for a sliding object)
And what if instead we use a disk of radius $R$ and mass $M$ that mounted on a massless shaft of radius $r \ll R$ and have the whole thing roll down an incline with a groove?

\[ m \sim 0 \]
And what if instead we use a disk of radius $R$ and mass $M$ that mounted on a massless shaft of radius $r \ll R$ and have the whole thing roll down an incline with a groove?

\[ \begin{align*}
Mg \sin \theta - f_s &= Ma_{CM} \\
&= M \frac{g \sin \theta}{1 + \frac{I_{CM}}{Mr^2}}
\end{align*} \]
$a_{CM} = \frac{g \sin \theta}{1 + \frac{I_{CM}}{Mr^2}}$

$I_{CM} = \frac{1}{2} MR^2$

$g \sin \theta$

Very small if $R \gg r$!